

# Higher order fermion effective interactions in a bosonization approach

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Received: date / Accepted: date

## Abstract

Three different fermion effective potentials given by series of bilinears,  $\sum_j^N (\bar{\psi}_a \psi_a)^{2j}$ ,  $\sum_j^N (\bar{\psi}_a \psi_a)^j$  and also  $\sum_j^N (\bar{\psi}_a \gamma_\mu \psi_a)^{2j}$  where  $a = 1, \dots, N_r$  and integer  $j$  are investigated by introducing sets of auxiliary fields. A minimal procedure is adopted to deal with the auxiliary fields and an effective bosonized model in each case is found by assuming weak field fluctuations, i.e. weak enough when compared to (normalized) coupling constants. Different fermion condensates are considered for the ground state in the first two series analysed and the factorization of all higher order condensates into the lowest order one is found in most cases, i.e. in general  $\langle (\bar{\psi}_a \psi_a)^n \rangle \propto \langle \bar{\psi}_a \psi_a \rangle^n$ . For the case of the third series built with vector-type bilinears no condensation is assumed to occur. The corresponding (weak) scalar fields effective models for the three cases are expanded in polynomial interactions. The resulting low energy effective boson model may exhibit new approximate symmetry depending on the terms present in the original series-model and on the values of the coupling constants.

## 1 Introduction

Higher order polynomial interactions usually appear in effective field theories including in cases in which non-polynomial interactions might be expanded into series of polynomial interactions [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. Although higher order interactions are usually irrelevant from a renormalization group analysis, one can ask whether and how they can contribute separately to the ground state and dynamics of the system. Further technical difficulties arise requiring reliable approximative methods to treat them. Higher order effective interactions contribute, in particular, in the framework of the Effective Field Theories (EFT) that has been shown to be suitable for the problem of (few) N-body states [12, 13]. The present work is to be seen as a non perturbative approach to these problems being that it decomposes the higher order effective fermion potential into N-body states. In this perspective, the factorization problem mentioned below may have consequences in few body systems since

some 3 and 4 body states are usually described only in terms of the 2 and 3 body scales and parameters [19]. Quantum Chromodynamics and the Electroweak theory are emblematic examples of theories whose effective models are of high interest for particular sectors of the corresponding phase diagrams. For instance, one of the most well known examples for higher order effective interactions comes from the low energy Quantum Chromodynamics (QCD) in which series of higher order couplings appear [14, 15, 16, 17]. Although the present work does not address QCD and their effective models, multifermion states and condensates ( $\langle \bar{\psi}_a \psi_a \rangle^n$ , for  $n = 1, 2, \dots$ ) usually considered in QCD can also be considered for different fermion models when a spontaneous symmetry breaking takes place. In QCD, due to difficulties associated with calculating higher order multiparticle states and condensates, it was proposed a factorization hypothesis of the higher order condensates into the lowest order one (i.e.  $\langle (\bar{q}q)^m \rangle \sim (\langle \bar{q}q \rangle)^m$ ) [20, 21]. However one finds it is not a very good approximation since  $\langle (\bar{q}q)^m \rangle \neq (\langle \bar{q}q \rangle)^m$  [22, 23]. This issue of higher order condensates, and their (non)factorization, might be however a general problem for fermion quantum field theories undergoing certain spontaneous symmetry breakings. One of the aims of the present work is to investigate the very low energy regime of general fermion effective models with higher order interactions whenever composite fermion states and condensates are formed. How and under which circumstances condensation occurs will not be discussed. Although these models are easily made invariant under  $U(1)$ ,  $U(1)^{N_r}$  and  $U(N_r)$ , being that all bilinears  $(\bar{\psi}_a \psi_a)$  have an implicit summation over  $a$ , the role of these symmetries will not be specifically investigated here. Further motivations for the present work can be found in systems where  $n$ -body states (mainly  $n = 2$  and 3 or 4) arise with some approximate degeneracy among composite states with different number of particles, in particular in cold atoms [24, 25] where (non relativistic) 2, 3 and 4 boson or fermion states are formed and in (light scalar) meson spectroscopy [26, 27] where (relativistic) light mesons with quark-antiquark and tetraquark structures have similar masses.

The non-perturbative auxiliary field method is known to provide good results for small fluctuations and weak coupling constant being directly extended to incorporate loop perturbative corrections [28]. In spite of the difficulties associated to a more complete and exact account of the nonlinearities [29] it has been widely and successfully applied to different models such as Gross Neveu, Nambu-Jona-Lasinio, among other models [30, 31, 32, 33, 8, 34]. This method can be implemented by means of shifts of the auxiliary fields to produce interactions that cancel out the original interactions of the model yielding a linearization of the original Lagrangian. A similar procedure given in Ref. [35] also produce an effective boson model for the original fermion model. In this work, and in Ref. [23], the auxiliary field method is considered to incorporate the corresponding higher order fermion composite states and condensates ( $\langle \bar{\psi}_a \psi_a \rangle^n$ ) by means of suitable shifts of extra (higher order) auxiliary fields. With the present approach however it is possible to envisage if these higher order condensates factorize or not into the lowest order one.

In this work, three different effective fermion models are investigated with interactions given by simple series of bilinears  $\sum_n (\bar{\psi}_a \psi_a)^{2n}$ ,  $\sum_n (\bar{\psi}_a \psi_a)^n$  and  $\sum_j (\bar{\psi}_a \gamma_\mu \psi_a)^{2j}$ , for  $n \leq N$  or  $j \leq N$ , and where  $a = 1 \dots N_r$  is an internal quantum number. This work only deals with finite  $N$  effective interactions, in particular with a maximum power  $N_{max} = 4 \times N_r$  in four spacetime dimensions, since for  $N > N_{max}$  there is no non trivial fermion effective interaction due to

the Grassmann algebra. These series can be considered to be simply toy or test models or effective models for more fundamental theories. Strictly speaking, a renormalization group (RG) evolution for the more fundamental theory generates all the missing bilinears in the first series:  $(\bar{\psi}\psi)^m$  with powers  $m < 2^n$ . At some energy scale it is possible that some of the generated terms are relatively small when compared to the others, i.e. some effective coupling constants (those of the terms  $g_{2^n}$ ) are relatively larger than the others, by considering they have different dimensions. Alternatively, one may simply consider the first series (with the terms  $g_{2^n}$ ) as a mathematical limit of the more general series and try to understand the role of the terms of the more general series that are not included. In these cases (models either realistic due to the original fundamental interactions or as a limiting mathematical case of the general model), it is legitimate to analyse a resulting effective model with such series of bilinears.

Therefore, as discussed above, the aims of this work are the following. Firstly, a dynamical framework will be considered to investigate if the higher order condensates factorize or not (whenever they can be formed) into the lowest order condensate, i.e. if they behave as  $< (\bar{\psi}_a \psi_a)^n > \propto < \bar{\psi}_a \psi_a >^n$  or not, up to  $n = N$  and for  $a = 1 \dots N_r$ . This is done by articulating further the non perturbative auxiliary field method to treat higher order interactions self consistently, hopefully it can bring some insight/information on the role of such effective interactions in the low energy dynamics. This is done by building N-body (composite boson) states (built from N-fermion states) taking into account (the corresponding) N-body effective interactions in a dynamical framework. Finally, we wish to extract information about the contribution of higher order fermion interactions for the low energy dynamics for models in which there is no invariance under chiral symmetry ( $U(N) \times U(N)$  or  $SU(N) \times SU(N)$ ), which is an important symmetry in low energy Chromodynamics. In this way it is possible to provide hints or results about resulting properties that are not due to chiral symmetry. To provide the cancelations of the interactions, shifts are performed in the normalized Gaussian integrals within the standard auxiliary field method. A minimal procedure is adopted which requires the minimum number of auxiliary fields and the minimum number of shifts, preventing the appearance of ambiguities. These fields are assumed to be weak when compared to the mean fields. and few ways of extending the validity of the results are shown, i.e. for non necessarily weak fields. yielding essentially unchanged results. The second aim, is to expand the resulting effective model for the auxiliary fields to show the structure of the resulting polynomial effective model. Comparison of the results from the first (or third) series of interactions with the more general second series will show an analytical example of an extra symmetry for the resulting effective boson model. The article is organized as follows. In the next section the method is presented for a series of interactions of the type  $\sum_n (\bar{\psi}_a \psi_a)^{2^n}$ , the ground state gap equations of auxiliary fields are shown and the (secondary level) polynomial effective model is derived. A way to lift the weak field approximation is presented in Sec 2.2. In Sec. 3 a more general series, of the type  $\sum_n (\bar{\psi}_a \psi_a)^n$ , is investigated within the same procedure of Section 2. In Sec. 4 a model with interactions typical from a momentum independent local limit of the effective potential obtained by the exchange of a vector field ( $\sum_j^N (\bar{\psi}_a \gamma_\mu \psi_a)^{2j}$ ) is considered without the formation of condensates (mean fields). In the final section there is a summary of the results.

## 2 Series of interactions $\Sigma_n(\bar{\psi}_a\psi_a)^{2^n}$

Consider the generating functional of an effective model for fermions  $Z = \int \mathcal{D}[\bar{\psi}_a, \psi_a] e^{i \int_x \mathcal{L}[\psi_a, \bar{\psi}_a]}$ , where  $\int_x$  denotes space-time integration in d-dimensions, with the Lagrangian density given by:

$$\mathcal{L} = \bar{\psi}_a(x) (i\not{\partial} - m_a) \psi_a(x) + \sum_n^N g_{2^n} (\bar{\psi}_a\psi_a)^{2^n}, \quad (1)$$

where  $g_{2^n}$  are the effective coupling constants with dimension:  $[g_{2^n}] = M^{d-(d-1)2^n}$ , where  $M$  has dimension of mass,  $m_a$  are the masses for each of the fermion species and the index  $a = 1 \dots N_r$  stands for the fermion components being that in each bilinear has an implicit summation over  $a$  and the mass term is therefore diagonal. The fermion interactions will be eliminated in favor of a set of scalar auxiliary fields which might give rise to the scalar structures of the type  $[(\bar{\psi}_a\psi_a)^n]$ .

These auxiliary fields (a.f.) are introduced by means of the following unity integrals multiplying the generating functional:

$$N' \int \mathcal{D}[\varphi_n] e^{-i \int_x \frac{1}{2} \sum_n^N \frac{1}{d_n} \varphi_n^2(x)} = 1, \quad (2)$$

where  $N'$  is a normalization constant and the parameters  $d_n$  are left free for the sake of generality. Alternatively a rescaled set of auxiliary fields could have been chosen:  $\frac{1}{d_n} \varphi_n^2 = \tilde{\varphi}_n^2$ , for  $\tilde{\varphi}_n = \frac{1}{\sqrt{d_n}} \varphi_n$  where the parameters could have been chosen to be simply  $d_n = 1$ . The necessary shifts of the a.f. needed to cancel out all the interactions can be made minimal shifts, i.e., the simplest shifts for the minimum number of auxiliary fields which do not introduce Lagrangian terms that were not presented in the original model. For the model of expression (1) the shifts are given by:

$$\varphi_n^2 \rightarrow (\varphi_n - \beta_n (\bar{\psi}_a\psi_a)^{2^{n-1}})^2, \quad (3)$$

where  $\beta_n$  are dimensionful parameters that are determined by imposing the corresponding cancelations of all polynomial interactions.

To obtain a finite number of equations, let us consider the series ends at  $n = N = 5$ , being easily generalized. The conditions for the cancelations of the polynomial interactions are the following:

$$\begin{aligned} g_{32} &= \frac{\beta_5^2}{2d_5}, \\ g_{16} &= -\frac{2\beta_5}{2d_5} \varphi_5 + \frac{\beta_4^2}{2d_4}, \\ g_8 &= -\frac{2\beta_4}{2d_4} \varphi_4 + \frac{\beta_3^2}{2d_3}, \\ g_4 &= -\frac{2\beta_3}{2d_3} \varphi_3 + \frac{\beta_2^2}{2d_2}, \\ g_2 &= -\frac{2\beta_2}{2d_2} \varphi_2 + \frac{\beta_1^2}{2d_1}. \end{aligned} \quad (4)$$

For an arbitrary  $n$ , these conditions can be written in the following form:

$$\begin{aligned} g_{2^n} &= -\frac{\beta_{n+1}}{d_{n+1}}\varphi_{n+1} + \frac{\beta_n^2}{2d_n}, & \text{for } n < N, \\ g_{2^n} &= \frac{\beta_n^2}{2d_n}, & \text{for } n = N. \end{aligned} \quad (5)$$

If the parameters  $\beta_n$  are then considered to be functions of different a.f. the above conditions one must guarantee that the shifts of the a.f. still have unity Jacobian. In fact all these shifts yield  $\beta_n = \beta_n[\varphi_{n+1}]$  and these still yield unity Jacobian. In fact, different shifts that could cancel out the fermion interactions would introduce other non linearities and the need of extra a.f. or non unity Jacobians. These parameters will assume numerical values (for fixed values for the coupling constants  $g_n$ ) when solving the gap equations, and then  $\beta_n \rightarrow \beta_n[\varphi_{n+1}^{(0)}]$ . It yields the following relations:

$$\begin{aligned} \beta_n &= \sqrt{2d_n \left( g_{2^n} + \frac{\beta_{n+1}}{d_{n+1}}\varphi_{n+1} \right)} & \text{for } n < N, \\ \beta_n &= \sqrt{2d_n g_{2^n}} & \text{for } n = N. \end{aligned} \quad (6)$$

In this case there is no ambiguity in the definitions of the parameters  $\beta_n$  as functions of the a.f.  $\beta_i[\varphi_{j+1}]$  written above. This minimal procedure is valid when all the fermion coupling constants, except  $g_{2^N}$ , are quite strong and (1) a subset of  $\varphi_{N-1}$  fields only assume positive values or (2) these auxiliary fields are weak with respect to the mean field which are weaker than the (normalized) fermion coupling constants. This means that higher order auxiliary fields, which are introduced to cancel out progressively more irrelevant fermion interactions, are progressively weaker, i.e.  $|\varphi_m \beta_m| < g_{2^{m-1}}$  (positive coupling constants) where  $\varphi_m$  is the mean field plus the fluctuation. Some ways to lift these conditions of weak field regime are provided in the next section.

The resulting effective action is given by:

$$S_{eff} = \int d^4x \left[ \bar{\psi}_a \left( i\not{\partial} - m_a + \frac{\beta_1}{d_1}\varphi_1 \right) \psi_a - \sum_{n=1}^N \frac{1}{2d_n}\varphi_n^2 \right]. \quad (7)$$

The saddle point equations for these auxiliary fields provide relations between the ground state average of the auxiliary fields  $\varphi_n$  and the progressively large powers of bilinears  $\langle (\bar{\psi}_a \psi_a)^n \rangle$ . To show these relations, consider that  $\frac{\delta \beta_1}{\delta \varphi_n} = \left( \prod_{i=2}^{n-1} \frac{\varphi_i}{\beta_i} \right) \frac{d_1}{d_n} \frac{\beta_n}{\beta_1}$ . It yields:

$$\frac{\langle \varphi_n \rangle}{\beta_n} \equiv \frac{\varphi_n^{(0)}}{\beta_n} \equiv \langle (\bar{\psi}_a \psi_a)^n \rangle. \quad (8)$$

The auxiliary fields analysed in this work are all scalars, i.e. they encapsulate the combinations of the fermion bilinears for each species, namely:  $\varphi_n \sim (\bar{\psi}_a \psi_a)^n$  for implicit summation over  $a$ . Therefore these auxiliary fields represent a sum of (physical) states.

By integrating out fermions, the following effective action is obtained:

$$S_{eff} = -iTr \log \left( i\cancel{\partial} - m_a + \frac{\beta_1}{d_1} \varphi_1 \right) - \sum_{n=1}^N \int_x \frac{\varphi_n^2}{2d_n}, \quad (9)$$

where  $Tr$  is the traces taken over all the internal indices and integration over space-time. According to expression (6), there is a dependence of  $\beta_1$  on all the fields  $\varphi_n$  through the parametric dependence of  $\beta_1$  on  $\beta_n$  (for  $n \neq 1$ ), i.e.:

$$\beta_1 = \beta_1[\varphi_2, \beta_2] \rightarrow \beta_1[\varphi_2[\varphi_3[\dots[\varphi_N]]]]. \quad (10)$$

Therefore  $\beta_1$  in the effective mass encodes the non linearities of the model.

The resulting mean field (homogeneous) GAP equations are the following:

$$\begin{aligned} \frac{\varphi_1}{d_1} &= -i \frac{\beta_1}{d_1} Tr \frac{1}{i\cancel{\partial} - m_a + \frac{\beta_1}{d_1} \varphi_1}, & \text{for } n = 1, \\ \frac{\varphi_n}{d_n} &= -i \frac{\varphi_1}{d_1} \frac{\partial \beta_1}{\partial \varphi_n} Tr \frac{1}{i\cancel{\partial} - m_a + \frac{\beta_1}{d_1} \varphi_1}, & \text{for } n > 1, \end{aligned} \quad (11)$$

where

$$\begin{aligned} \frac{\partial \beta_1}{\partial \varphi_2} &= \frac{d_1}{\beta_1} \frac{\beta_2}{d_2}, \\ \frac{\partial \beta_1}{\partial \varphi_3} &= \frac{1}{2\beta_1} \frac{2d_1}{d_2} \frac{\partial \beta_2}{\partial \varphi_3} = \frac{d_1}{\beta_1} \frac{\beta_3}{d_3} \varphi_2, \\ &\dots \\ \frac{\partial \beta_1}{\partial \varphi_n} &= \left( \prod_2^{n-1} \frac{\varphi_i}{\beta_i} \right) \frac{d_1}{d_n} \frac{\beta_n}{\beta_1}. \end{aligned} \quad (12)$$

Therefore the gap equations can be written as:

$$\begin{aligned} \frac{\varphi_1}{d_1} &= \frac{\beta_1}{d_1} I_\Lambda, & \text{for } n = 1, \\ \frac{\varphi_n}{d_n} &= \frac{\beta_n}{d_n \beta_1} \left( \prod_{i=2}^{n-1} \frac{\varphi_i}{\beta_i} \right) I_\Lambda, & \text{for } n > 1, \end{aligned} \quad (13)$$

Where the following quantity was defined:

$$I_\Lambda = -iTr \frac{1}{i\cancel{\partial} - m_a^*}, \quad (14)$$

where the effective mass is given by:  $m_a^* = m_a - \frac{\beta_1}{d_1} \varphi_1^{(0)}$  being written in terms of the vacuum expected value of the auxiliary fields.  $\varphi_i^{(0)} = \langle \varphi_i \rangle$ .

Therefore for all the gap equations we can write:

$$\frac{\varphi_n}{\beta_n} = \frac{\varphi_1}{\beta_1} \left( \prod_{i=1}^{n-1} \frac{\varphi_i}{\beta_i} \right), \quad (15)$$

Table 1: Approximated dimensionless parameters and resulting variables from equations (6) and (11) for  $N_r = 2$ ,  $N = 3$ ,  $\tilde{M} = 0.9$  and  $\tilde{\Lambda} = 4$ .  $\tilde{\varphi}_2^{(0)}$  and  $\tilde{\varphi}_3^{(0)}$  are unambiguously determined by expressions (15).

$\tilde{g}_2$	$\tilde{g}_4$	$\tilde{g}_8$	$\tilde{\beta}_1$	$\tilde{\varphi}_1^{(0)}$	$M^*/M$	$\varphi_1^{(0)} = \tilde{\varphi}_1^{(0)}\beta_1$	$\tilde{\beta}_2$	$\tilde{\beta}_3$
$10^{-5}$	$10^{-5}$	$10^{-5}$	0.0	0.60	1.00	0.0	0.004	0.004
0.0001	0.001	0.001	0.15	0.65	1.02	0.19	0.055	0.044
0.01	0.001	0.001	0.15	0.65	1.02	0.19	0.055	0.045
0.01	0.01	0.01	0.20	0.66	1.03	0.13	0.18	0.14
0.02	0.01	0.01	0.23	0.64	1.05	0.15	0.18	0.14
0.03	0.01	0.01	0.29	0.66	1.06	0.19	0.18	0.14
0.1	0.01	0.01	0.47	0.70	1.18	0.33	0.18	0.14
1	0.01	0.01	1.46	1.02	3.41	1.49	0.25	0.14

This means that, for the model (1), all the higher order condensates are factorized into the lowest order condensate,  $\langle \bar{\psi}_a \psi_a \rangle$ . Therefore if the first gap equation has non trivial solution(s), all the solutions for the others are obtained. However, for this gap equation all the set of coupled algebraic expressions (6) for  $\beta_n$  must be solved together. This system turns out to be highly non linear and complicated. For these equations, all the variables and parameters were rescaled by an arbitrary constant of dimension of mass,  $\mu$ , such that  $g_{2n} = \tilde{g}_{2n}(\mu^{4-3.2^n})$ ,  $m_a = M = \tilde{M}\mu$ ,  $\varphi_n = \tilde{\varphi}_n\mu^2$  and  $\beta_n = \tilde{\beta}_n(\mu^{2-3.2^{n-1}})$ . Besides that, momentum is also rescaled by  $k = \tilde{k}\mu$ , and by performing the momentum integration with a covariant Euclidean cutoff  $\Lambda$ , it rescales to  $\Lambda = \tilde{\Lambda}\mu$ . The set of equations (6) and also (11,15) become independent of  $\mu$ . For lower dimensions, it should appear a maximum number number of components  $N_r$  for which the gap equations (in particular for  $\varphi_1$ ) provide non trivial results whereas for higher dimensions this issue should be less restrictive. In the next section this model will be expanded by considering three a.f., i.e.  $N = 3$ .

In Table 1 there are few numerical solutions for the case of equal masses  $\tilde{m}_a = \tilde{M} = 0.9$ ,  $N = 3$  and  $N_r = 2$  in four dimensions with  $\tilde{\Lambda} = 4$ . With  $N_r = 2$  it is possible to consider non trivial polynomial interactions up to  $(\bar{\psi}_a \psi_a)^{2^n=8}$ , i.e. for  $n = 3$ . This regime of the phase diagram shows explicitly the validity of the minimal procedure, i.e. within the weak field approximation, since  $\beta_2\varphi_2$  can fluctuate smoothly around the (positive or negative) vacuum value provided  $g_1 > |\beta_2\varphi_2|$ . The same for  $\beta_3\varphi_3$  with respect to  $g_4$ . Therefore expressions (6) have real solutions for weak fluctuations. The same analysis applies for all  $\beta_n$  ( $n < N$ ). It is interesting to note that, the condensates go to zero for larger coupling constant  $\tilde{g}_2$  because the other coupling constants were kept constants. It can be noted, analysing  $\tilde{\varphi}_1^{(0)}$  for the third, fourth, fifth and sixth lines, that to obtain the usual symmetric ground state (where  $M^*/M = 1$ ), the order in which the coupling constants are set to zero might be relevant.

## 2.1 Expansion of the model

In the following a large fermion mass (zeroth order) derivative expansion of the determinant is done such as to write down an effective polynomial model for the scalar fields. The fermion

determinant can be written as:

$$Tr \ln \left[ 1 + D_a \left( \beta_1 \frac{\varphi_1}{d_1} \right) \right] + Tr \ln D_a^{-1}, \quad (16)$$

where  $D_a = \frac{1}{i\phi - m_a^*}$ . The first terms expansion corresponds to:

$$S_{eff} \simeq S_{eff,(0)}[\varphi_i^{(0)}] + \sum_i^N \sum_j^N \frac{1}{n_i! n_j!} \int_{x_1, x_2} \frac{\delta^2 S_{eff}}{\delta \varphi_i(x_1) \delta \varphi_j(x_2)} \Big|_{\varphi_i = \varphi_i^{(0)}} \varphi_i(x_1) \varphi_j(x_2) + h.o.,$$

where  $\int_{x_1, x_2} = \int dx_1 \int dx_2$ , *h.o.* stands for (even) higher order derivatives,  $n_i, n_j = 0, 1, 2$  are such that  $n_i + n_j = 2$ . The first derivative term is set to zero due to the stability condition. A constant multiplicative factor appears for each of the derivative and therefore a field redefinition can be done to simplify the resulting expressions. These field redefinitions are given by:

$$\varphi_1 \rightarrow \varphi_1 \frac{\beta_1}{d_1} \equiv \chi_1, \quad \varphi_2 \rightarrow \varphi_2 \left( \frac{\varphi_1^{(0)}}{\beta_1} \frac{\beta_2}{d_2} \right) \equiv \chi_2, \quad \varphi_3 \rightarrow \varphi_3 \left( \frac{\varphi_1^{(0)}}{\beta_1} \frac{\varphi_2^{(0)}}{\beta_2} \frac{\beta_3}{d_3} \right) \equiv \chi_3, \quad \dots \quad (17)$$

With this redefinition, all the auxiliary fields  $\chi_i$  will have the same dimension. By assembling the interaction terms, it yields for the first four auxiliary fields:

$$\begin{aligned} \mathcal{V}_I^{eff} = & \frac{1}{2} \left[ (c_2 + c_{2,1}) \chi_1^2 + (c_2 + c_{2,2}) \chi_2^2 + (c_2 + c_{2,3}) \chi_3^2 \right] + \sum_{n \geq 3} [c_n \chi_1^n + (c_n + c_{n,2}) \chi_2^n + \\ & (c_n + c_{n,2} + c_{n,3}) \chi_3^n] + \sum_{i,j,k} t_{i,j,k} \chi_1^i \chi_2^j \chi_3^k, \end{aligned} \quad (18)$$

where  $t_{i,j,k}$  are defined for  $i + j + k = m \geq 2$  being  $i, j, k = 0, \dots, m$  where at least two indices are different from zero, and where  $c_n$  and  $c_{n,m}$  are the resulting self interaction coupling constants and contributions for masses, and also the couplings  $t_{i,j,k}$  are those couplings between at least two different components, being that at least two indices are non zero, i.e.  $i, j \neq 0$  or  $i, k \neq 0$  and so on.  $c_2$  are the terms provinient from the unity integrals of the auxiliary fields.

In the limit of same masses for all the fermion components ( $m_a = m$ ) the same kernels  $D_a = D$  are obtained. Should the Lagrangian fermion masses of each of the components be different there would be a degree of freedom more in the kernels  $D \rightarrow D_a$  with which the arguments below can be drawn although not necessarily yielding the same conclusions. This allows the resulting coupling constants to be written and defined in an uniform notation. They can be written as:

$$\begin{aligned} c_2 &= Tr D^2, \quad c_{2,1} = \frac{d_1^2}{\beta_1^2} \\ c_{2,2} &= \frac{d_1}{\beta_1 \varphi_1^{(0)}} Tr D + \frac{d_2 \beta_1^2}{\beta_2^2}, \\ c_{2,3} &= \frac{d_2 \beta_1}{\beta_2 \varphi_1^{(0)} \varphi_2^{(0)}} Tr D + \frac{\beta_1^2 \beta_2^2 d_3}{\varphi_1^{(0)2} \varphi_2^{(0)2} \beta_3^2}, \end{aligned} \quad (19)$$



The couplings  $t_{i,j,k}$  are due to the parametric dependence of the term  $\beta_1\varphi_1$  of expression (16) on the other fields,  $\beta_1\varphi_1 \rightarrow \beta_1[\varphi_2, \beta_2]\varphi_1 \rightarrow \beta_1[\varphi_2, \dots, \varphi_N]\varphi_1$ . The massive terms of the auxiliary fields cannot be equal according to the above results however they might be nearly equal if the contributions  $c_{2,n}$  are progressively smaller, i.e. for progressively large  $\beta_n$  for larger  $n$ : ( $\beta_3^2 > \beta_2^2 > \beta_1^2$ ) and/or  $|\varphi_n^{(0)}| \ll |\varphi_{n+1}^{(0)}|$ . Although this limit could be spoiled by the weak field condition observed for expressions (6) but it is not in all the cases analysed in this section. Even if the field redefinition above is not done, the same limit is achieved for the a.f.  $\varphi_n$ 's the case in which  $\beta_n^2/\beta_{n-1}^2$  is small what is achieved nearly in the same regime as the progressively large condensates regime. For the results of Table (1) the weakest coupling constants  $g_2$  (largest condensate values) correspond nearly to values in which the coefficients  $c_{2,n}$  become smaller and the limit below of approximated symmetric effective potential is valid. For the second order interactions between the different components  $\chi_i\chi_j$ , given by the terms  $t_{i,j,k}$  of expression (18), it yields:

$$t_{1,1,0} = t_{1,0,1} = t_{0,1,1} - t'_2, \quad (20)$$

where

$$\begin{aligned} t_{1,1,0} &= -\frac{d_1}{\beta_1\varphi_1^{(0)}} Tr D_a + Tr D_a^2, \\ t'_2 &= \frac{d_2\beta_1}{\varphi_1^{(0)}\varphi_2^{(0)}} Tr D_a, \end{aligned} \quad (21)$$

All these couplings have the same dimension, in  $d=4$  they have dimension mass square. For progressively higher order interaction terms, different structures appear for higher order auxiliary fields  $\chi_4, \chi_5$  and so on. In this case, as well as in higher order interactions, there is a privileged role of the first component  $\varphi_1$  (now  $\chi_1$ ) over the others generating a sector of the model of higher symmetry than the full original model. This different role for the lowest order fields, in particular  $\varphi_1$ , is more apparent and explicitly in the case mean fields are zero. These second order terms might become equal by adjusting the coupling constants of the original model, defined in expression (1), and consequently the parameters  $\beta_i$  and  $d_i$ , such that it could yield instead:  $t_{0,1,1,0} - t'_2 = t_{0,1,0,1} - t'_2 = 0$ . Also, in the same limit of progressively larger values of the condensates  $\varphi_n^{(0)} \ll \varphi_{n+1}^{(0)}$  mentioned above, the differences in the coupling constants become smaller.

The expansion at third order, for which  $(i+j+k=3)$  with at least two indices non zero), also yields terms with identical coefficients and terms with slightly different coefficients. It can be written that:

$$t_{2,1,0} = t_{2,0,1} = t_{1,2,0} = t_{1,0,2} = t_{0,2,1} - t_3 = t_{0,1,2} - t'_3, \quad (22)$$

where  $t_3 - t'_3 \sim \varphi_1^{(0)}/\beta_1$ , which is small with respect to  $t_{2,1,0}$  and other terms in the limit of small  $\varphi_1^{(0)}/\beta_1$  considered above. It appears an approximated identification for all the couplings of the type:

$$t_{2,1,0} \sim t_{2,0,1} \sim t_{0,2,1} \sim t_{1,2,0} \sim t_{0,1,2} \dots \quad (23)$$

This is valid also for the masses in expression (18) and all the higher order couplings. Finally the term  $\varphi_1\varphi_2\varphi_3$  has a coefficient  $t_{1,1,1}$  whose difference  $t_{1,1,1}-t_{1,2,0} \sim \beta_n/\beta_{n-1}$  and  $t_{1,1,1}-t_{1,2,0} \sim \varphi_{n-1}^{(0)}/\varphi_n^{(0)}$ . By considering that there is a factor 3 with respect to the effective interactions of the type  $\varphi_n^3$  from the combinatorial factor in the expansion, in this limit, it yields in general the following effective potential:

$$\mathcal{V}_{eff}^{large \varphi_i^{(0)}} \simeq \sum_{n=2} g_n (\chi_1 + \chi_2 + \chi_3 + \dots)^n. \quad (24)$$

The expansion also provides kinetic terms which appear by writing the kernels  $D$  with a part diagonal in momentum space and another part diagonal in coordinate space [36], i.e.  $D = -(i\not{\partial} + m^*) \cdot S_0$ , where  $S_0 = 1/(k^2 + m^{*2})$ . The lowest order derivative terms in the limit considered above for the expression (24), is the following:

$$\Delta\mathcal{L}_{eff} = \frac{F}{2} \partial_\mu (\chi_1 + \chi_2 + \chi_3 + \dots) \partial^\mu (\chi_1 + \chi_2 + \chi_3 + \dots),$$

where  $F$  is a constant to be calculated from the expansion. This effective boson model is invariant under any transformation that keeps the length  $(\sum_i \chi_i)$  invariant. One set of continuous transformations is given by:

$$\begin{aligned} \chi_1 &\rightarrow \chi_1 + b_1\chi_2 + c_1\chi_3, \\ \chi_2 &\rightarrow \chi_2 + a_2\chi_1 - c_1\chi_3, \\ \chi_3 &\rightarrow \chi_3 - a_2\chi_1 - b_1\chi_2. \end{aligned} \quad (25)$$

For this transformation considering three fields,  $N=3$ , there are three parameters in the transformations. For the case of  $N$  fields there will have  $N^2 - 2N$  parameters/generators of the algebra. The resulting particle excitations present therefore the same mass, since these expressions already correspond to fluctuations dynamics. The resulting algebra for this set of transformations will not be discussed here. Apart from the above symmetry this effective model is invariant under simple permutations of the fields such as  $(\chi_1 \rightarrow \chi_2, \chi_2 \rightarrow \chi_3 \text{ and } \chi_3 \rightarrow \chi_1)$ .

## 2.2 Removing weak fields conditions

Two ways of overcoming the limitation of weak field are given in this Section. Although they might require a non minimal procedure they yield the same result as shown above. However one might also simply require the scalar fields  $\varphi_n$ , for  $n < N$ , to only assume positive values in which case the dynamics would be resctricted to one side of the effective potential  $V_{eff}(\varphi_i > 0)$ .

The first more general solution is to introduce further set of auxiliary fields  $\xi_n$  with the same shifts provided above with different parameters  $\beta'_n$ . In this case expression (6) can be rewritten as:

$$\begin{aligned} \beta_n^2 + \beta_n'^2 &= 2d_n \left( g_{2^n} + \frac{\beta_{n+1}}{d_{n+1}} \varphi_{n+1} + \frac{\beta_{n+1}'}{d_{n+1}} \xi_{n+1} \right) \quad \text{for } n < N, \\ \beta_n^2 + \beta_n'^2 &= 2d_n g_{2^n} \quad \text{for } n = N. \end{aligned} \quad (26)$$

It looks the number of free parameter now is doubled. However to avoid limiting to weak field,  $\beta'_n$  is associated to the imaginary part of the functions  $\beta_n$  being that the corresponding field  $\xi_n$  might not correspond necessarily to a physical degree of freedom to avoid double counting.

A second, more general, solution to overcome the eventual limited range of values of the auxiliary fields is to consider vector parameters for the shifts in the Gaussian integrals (51). For the sake of the argument, let us consider the four dimensional case with  $d_n = 1$  to provide an example. The a.f. in the unity Gaussian integral can be written as:

$$\begin{aligned} -\frac{\varphi_1^2}{2} &\rightarrow -\frac{1}{2A^\mu A_\mu} \left( A^\mu \varphi_1 - C_1^\mu \bar{\psi} \psi \right) \left( A_\mu \varphi_1 - C_\mu^1 \bar{\psi} \psi \right), \\ -\frac{\varphi_2^2}{2} &\rightarrow -\frac{1}{2A^\mu A_\mu} \left( A^\mu \varphi_2 - C_2^\mu (\bar{\psi} \psi)^2 \right) \left( A_\mu \varphi_2 - C_\mu^2 (\bar{\psi} \psi)^2 \right), \\ -\frac{\varphi_3^2}{2} &\rightarrow -\frac{1}{2A^\mu A_\mu} \left( A^\mu \varphi_3 - C_3^\mu (\bar{\psi} \psi)^4 \right) \left( A_\mu \varphi_3 - C_\mu^3 (\bar{\psi} \psi)^4 \right), \end{aligned} \quad (27)$$

where  $A^\mu$  is constant. Only two non zero components are enough, i.e.  $A_\mu = (A_0, A_1, 0, 0)$  and the reason is that it must have the minimum number of degree of freedom (arbitrary constants to be determined below) and it should account for the possibility of non trivial contractions  $C_\mu^n \cdot A^\mu$  (where  $C_\mu^n$  have two components as well). In any case  $A_\mu$  can be chosen so that:  $A_0 = A_1 \sqrt{2}$  and then it can be written in terms of only one free parameter  $A_1$ . The vectors  $C_\mu^n$  are parameters that might be functions of the auxiliary fields in the same way the parameters  $\beta_n$  of the minimal procedure do. Also it is enough to have two non zero components, as shown below, to allow for positive and negative normalization  $C_n^\mu \cdot C_\mu^n$ . With these shifts the cancelations are obtained with the following relations:

$$\begin{aligned} \frac{C_3^\mu \cdot C_\mu^3}{A_\mu A^\mu} &= 2g_8, \\ \frac{C_2^\mu \cdot C_\mu^2}{A_\mu A^\mu} &= 2 \left( g_4 + \frac{C_\mu^3 \cdot A^\mu}{A^\mu \cdot A_\mu} \varphi_3 \right), \\ \frac{C_\mu^1 \cdot C_1^\mu}{A_\mu A^\mu} &= 2 \left( g_2 + \frac{C_\mu^2 \cdot A^\mu}{A^\mu \cdot A_\mu} \varphi_2 \right), \end{aligned} \quad (28)$$

being that  $N - 1$  of these vector parameters  $C_\mu$  become functions of some of the auxiliary fields (in the case of  $N = 3$  they are  $C_2^\mu$  and  $C_1^\mu$ ). It yields an effective action with the same shape and structure of expression (9), given by:

$$S_{eff} = -iTr \ln \left( i\not{\partial} - m_a + g_1 + \frac{C_1^\mu A_\mu \varphi_1}{A^\mu A_\mu} \right) - \int_x \left( \frac{\varphi_1^2}{2} + \frac{\varphi_2^2}{2} + \frac{\varphi_3^2}{2} \right). \quad (29)$$

The difficulty with this parameterization might be the number of free parameters ( $A^\mu$  and  $C_\mu^n$ ) that is larger than the number of expressions (28). However this can be avoided by a direct identification with the minimal procedure which can be given if:

$$\beta_n^2 = C_\mu^n C_\mu^\mu, \quad \text{and} \quad A^\mu \cdot A_\mu = 1. \quad (30)$$

Now, the case of negative  $\beta_n^2$  corresponds to the negative normalization of the vector  $C_n^\mu C_\mu^n$ , consequently  $\varphi_n$  ( $\chi_n$ ) do not need to assume complex values. Therefore further choices for the parameters can be done, such as:  $C_3^\mu = (L_0, 0, 0, 0)$  for positive  $g_8$ ,  $A^\mu = (\sqrt{2}, 1, 0, 0)$  for the normalization of the Gaussian integrals,  $C_2^\mu = (K_2, D_2, 0, 0)$  and  $C_1^\mu = (K_1, D_1, 0, 0)$ . With this, the second and third expressions of (28) could be expected to fix four undetermined parameters ( $K_1, K_2, D_1$  and  $D_2$ ). The only choice that makes these expressions non ambiguous is that  $K_i = 0$  and  $D_i \neq 0$  if  $C_\mu^i C_i^\mu < 0$  and  $K_i \neq 0$  and  $D_i = 0$  if  $C_\mu^i C_i^\mu \geq 0$ . Therefore to eliminate the ambiguity in defining the components of  $C_\mu^n$ :  $K_1$  and  $K_2$  become functions or parameters that parameterize only the positive values of  $C_\mu^n C_n^\mu$ , and  $D_1$  and  $D_2$  are functions or parameters that parameterize only the negative values of  $C_\mu^n C_n^\mu$ . Second and third conditions (28) can then be written in the two cases of positive or negative arguments as:

$$\begin{aligned} K_i^2 &= 2[g_{2^i} + (K^{i+1}\sqrt{2} - D^{i+1})\varphi_{i+1}] \geq 0, \\ -D_i^2 &= 2[g_{2^i} + (K^{i+1}\sqrt{2} - D^{i+1})\varphi_{i+1}] < 0. \end{aligned} \quad (31)$$

### 3 More general series $\Sigma_n(\bar{\psi}_a \psi_a)^n$

In this section we consider a more general series of bilinears. The generating functional will be given by:

$$Z = \int \mathcal{D}[\bar{\psi}_a, \psi_a] e^{i \int_x \mathcal{L}[\psi_a, \bar{\psi}_a]}, \quad (32)$$

where a general series of bilinears of the following form will be considered:

$$\mathcal{L} = \bar{\psi}_a(x) (i\not{\partial} - m_0) \psi_a(x) + \sum_{n=2}^N g_{2n} (\bar{\psi}_a \psi_a)^n, \quad (33)$$

where the case of even  $N$  will be addressed, and the case for  $N$  odd will be discussed below shortly. In each bilinear there is an implicit sum over  $a$  and the mass term is therefore diagonal.

The auxiliary fields are introduced by means of  $N/2$  unity integrals that are given by:

$$N' \int \mathcal{D}[\xi_m] e^{-i \int_x \sum_m^{N/2} \frac{1}{2d_m} \xi_m^2(x)} = 1, \quad (34)$$

where  $d_m$  are constants, eventually they can be set to 1.

The simplest necessary shifts of the auxiliary fields that cancel out the interactions can be written as:

$$\frac{1}{2d_m} \xi_m^2 \rightarrow \frac{1}{2d_m} \left( \xi_m - B_m (\bar{\psi}_a \psi_a)^m - A_m (\bar{\psi}_a \psi_a)^{m-1} \right)^2. \quad (35)$$

There are other possible shifts in the auxiliary fields, however these are the simplest ones that cancel out all the polynomial interactions.

The cancelation of all the interactions occur if the following relations hold:

$$\begin{aligned}
g_4 &= \frac{B_1^2}{2d_1} + \frac{A_2^2}{2d_2} - \frac{\xi_3 A_3}{d_3} - \frac{\xi_2 B_2}{d_2}, \\
g_6 &= \frac{A_2 B_2}{d_2} - \frac{B_3}{d_3} \xi_3 - \frac{A_4}{d_4} \xi_4, \\
g_8 &= \frac{B_2^2}{2d_2} + \frac{A_3^2}{2d_3} - \frac{B_4}{d_4} \xi_4, \\
g_{10} &= \frac{B_3 A_3}{d_3}, \\
g_{12} &= \frac{B_3^2}{2d_3} + \frac{A_4^2}{2d_4} \\
g_{14} &= \frac{B_4 A_4}{d_4} \\
g_{16} &= \frac{B_4^2}{2d_4} \\
&\dots \\
g_{2n} &= \frac{B_{n/2}^2}{2d_{n/2}} + \frac{A_{(n+2)/2}^2}{2d_{(n+2)/2}} - \frac{B_n}{d_n} \xi_n - \frac{A_{n+1}}{d_{n+1}} \xi_{n+1} \\
&\quad \text{n even, } n \leq N-1.
\end{aligned} \tag{36}$$

In particular for  $n = N$

$$g_{2N} = \frac{B_{N/2}}{2d_{N/2}}. \tag{37}$$

All the discussion and remarks made in the last section applies here for the case of enforcing the weak field conditions or to lift them.

Since one of the aims of this calculation is to show the structure of the resulting model for auxiliary fields, and to compare with the model from the last section, the series will stop at  $N = 6$ , i.e.  $(\bar{\psi}\psi)^6$ , such that 3 auxiliary fields are needed. According to the expressions above, the parameters of the shifts  $A_n, B_n$  must be considered to be field dependent. This dependence has an unique possible choice which is given by:

$$\begin{aligned}
B_1[\xi_2, \xi_3] &= \sqrt{(2d_1) \left( g_4 - \frac{A_2^2}{2d_2} + \frac{\xi_3 A_3}{d_3} + \frac{\xi_2 B_2}{d_2} \right)} \\
A_2[\xi_3, \xi_4] &= \frac{d_2}{B_2} \left( g_6 + \frac{B_3}{d_3} \xi_3 \right), \\
B_2[\xi_4, \xi_5] &= \sqrt{(2d_2) \left( g_8 - \frac{A_3^2}{2d_3} \right)}, \\
A_3[\xi_5, \xi_6] &= \frac{d_3}{B_3} g_{10}, \\
B_3[\xi_6, \xi_7] &= \sqrt{2d_3 g_{12}}.
\end{aligned} \tag{38}$$

In general, for  $N$  even:

$$\begin{aligned}
B_{N/2} &= \sqrt{g_{2n} 2d_{n/2}} \quad n = N \text{ even}, \\
B_n &= \sqrt{(2d_n)(g_{4n} + \Xi)}, \quad n \leq N-1 \text{ even}, \\
A_n &= \frac{d_n}{B_n} \left( g_{2(n+1)} + \frac{B_{n+1}}{d_{n+1}} \xi_{n+1} + \frac{A_{n+2}}{d_{n+2}} \xi_{n+2} \right) \quad n \geq 2,
\end{aligned} \tag{39}$$

where

$$\Xi = -\frac{A_{n+1}^2}{2d_{n+1}} + \frac{B_{n+1}}{d_{n+1}} \xi_{n+1} + \frac{A_{n+2}}{d_{n+2}} \xi_{n+2}$$

and therefore:  $B_n = B_n[\xi_n, \xi_{n+1}, \xi_{n+2}]$ . The limit of weak field is also assumed and the corresponding discussion of the last section applies here.

The action can be rewritten as:

$$S_{eff} = \int_x \left[ \bar{\psi}_a (i\partial - (\tilde{M})) \psi_a - \sum_m^{N/2} \frac{1}{2d_m} \xi_m^2(x) - \frac{A_1(A_1 - 2\xi_1)}{2d_1} \right], \tag{40}$$

where

$$\tilde{M} = m_a - \frac{(B_1 \xi_1 - B_1 A_1)}{d_1} - \frac{A_2}{d_2} \xi_2$$

Therefore  $A_1, B_1$  encode all the non linearities of the model. The saddle point equations for this model provides relations between the ground state average of the auxiliary fields  $\xi_n$  and the progressively large power of bilinears:  $\langle (\bar{\psi}_a \psi_a)^n \rangle$ . In the same way it was done in last Section, with expressions (8), one has the following relation between the ground state averaged value of the a.f. and the composite fermion condensates:

$$\frac{\xi_n^{(0)}}{B_n} \equiv \langle (\bar{\psi}_a \psi_a)^n \rangle + \frac{A_n}{B_n} \langle (\bar{\psi}_a \psi_a)^{n-1} \rangle. \tag{41}$$

With the integration of fermions the remaining terms, neglecting an irrelevant constant, are the following:

$$S_{eff} = -i \text{Tr} \ln \left( i\partial - m_a + \frac{(B_1 \xi_1 + B_1 A_1)}{d_1} + \frac{A_2}{d_2} \xi_2 \right) - \int_x \sum_m^{N/2} \frac{\xi_m^2(x)}{2d_m}. \tag{42}$$

where  $C_1 = \frac{A_1(A_1 - 2\xi_1)}{2d_1}$ .

The gap equations for the homogeneous a.f. are therefore the following:

$$\begin{aligned}
\frac{\xi_1}{d_1} &= \frac{A_1}{d_1} - \frac{B_1}{d_1} i \text{Tr} \frac{1}{i\partial - m_a^*}, \\
\frac{\xi_2}{d_2} &= - \left( \frac{A_2}{d_2} + \frac{\xi_1}{d_1} \frac{d_1 B_2}{B_1 d_2} \right) i \text{Tr} \frac{1}{i\partial - m_a^*}, \\
\frac{\xi_3}{d_3} &= \left[ \left( \frac{\xi_1}{d_1} + \frac{A_1}{d_1} \right) \frac{2d_1}{2B_1} \left( \frac{A_3}{d_3} - \frac{d_2 B_3}{B_2 d_3} \frac{A_2}{d_2} \right) - \frac{\xi_2}{d_2} \frac{d_2 B_3}{B_2 d_3} \right] i \text{Tr} \frac{1}{i\partial - m_a^*},
\end{aligned} \tag{43}$$

where

$$I_\Lambda = -i \text{Tr} \frac{1}{i\not{D} - m_a^*}$$

and where the effective mass has been defined as:

$$m^* = m_a - \frac{(B_1 \xi_1^{(0)} - B_1 A_1)}{d_1} - \frac{A_2}{d_2} \xi_2^{(0)}. \quad (44)$$

In this expression, the fields are the homogeneous, mean field, solutions of the gap equations. Although the effective mass depends explicitly only on the first two a.f., the parameters  $A_1, B_1, A_2$  depend on the higher order a.f. as shown in expressions (38-39). At the end, all of the a.f.  $\xi_n$  contribute for the effective mass. The first three gap equations can be rewritten as:

$$\begin{aligned} \frac{\xi_1^{(0)}}{B_1} &= \frac{A_1}{B_1} + I_\Lambda, \\ \frac{\xi_2^{(0)}}{B_2} &= \left( \frac{A_2}{B_2} + \frac{\xi_1^{(0)}}{B_1} \right) \left( -\frac{A_1}{B_1} + \frac{\xi_1^{(0)}}{B_1} \right), \\ \frac{\xi_3^{(0)}}{B_3} &= \left[ \left( \frac{\xi_1^{(0)}}{B_1} - \frac{A_1}{B_1} \right) \left( \frac{A_3}{B_3} - \frac{A_2}{B_2} \right) - \frac{\xi_2}{B_2} \right] \left( -\frac{A_1}{B_1} + \frac{\xi_1^{(0)}}{B_1} \right), \end{aligned} \quad (45)$$

with the corresponding definitions of the parameters (functions)  $A_i$  and  $B_i$ . By writing these expressions in terms of all the higher order fermion condensates, for the general case  $A_1 \neq 0$ , one obtains:

$$\begin{aligned} \langle (\bar{\psi}_a \psi_a)^2 \rangle &= (\langle \bar{\psi}_a \psi_a \rangle)^2 - \frac{A_1}{B_1} \left[ 3 \langle \bar{\psi}_a \psi_a \rangle + 2 \left( \frac{A_2}{B_2} - \frac{A_1}{B_1} \right) \right], \\ \langle (\bar{\psi}_a \psi_a)^3 \rangle &= \langle \bar{\psi}_a \psi_a \rangle \left[ \frac{A_3}{B_3} \left( \langle \bar{\psi}_a \psi_a \rangle + \frac{A_1}{B_1} \right) + \langle (\bar{\psi}_a \psi_a)^2 \rangle - \frac{A_1 A_2}{B_1 B_2} \right] \\ &\quad - \frac{A_3}{B_3} \langle (\bar{\psi}_a \psi_a)^2 \rangle. \end{aligned} \quad (46)$$

From these expressions we conclude that if  $A_1 = 0$  there is a complete factorization of higher order condensates into the lowest order fermion-antifermion, i.e.:

$$\langle (\bar{\psi}_a \psi_a)^n \rangle = (\langle \bar{\psi}_a \psi_a \rangle)^n. \quad (47)$$

The case in which  $A_1 \neq 0$  corresponds to a constant shift of the first auxiliary variable,  $\xi_1$ , which is associated to the lowest order condensate  $\langle \bar{\psi} \psi \rangle$  and therefore to its redefinition.

For  $N$  odd, two of the shifts above (35) would receive contribution another term. The shifts for these two higher order auxiliary fields (i.e.  $(N-1)/2$  and  $(N-1)/2 - 1$ , being now the highest order a.f. is  $(N-1)/2$ ) must be modified to the following:

$$\frac{\xi_{(N-1)/2}^2}{d_{(N-1)/2}} \rightarrow \frac{1}{d_{(N-1)/2}} \left( \xi_{(N-1)/2} - B_{(N-1)/2} (\bar{\psi}_a \psi_a)^{(N-1)/2} \right)$$

$$\begin{aligned}
& -A_{(N-1)/2}(\bar{\psi}_a\psi_a)^{(N-1)/2-1} - C_{(N-1)/2}(\bar{\psi}_a\psi_a)^{(N-1)/2+1})^2, \\
\frac{\xi_{(N-1)/2-1}^2}{d_{(N-1)/2-1}} & \rightarrow \frac{1}{d_{(N-1)/2-1}} \left( \xi_{(N-1)/2-1} - B_{(N-1)/2-1}(\bar{\psi}_a\psi_a)^{(N-1)/2-1} \right. \\
& \left. - A_{(N-1)/2-1}(\bar{\psi}_a\psi_a)^{\frac{N-1}{2}-2} - C'_{(N-1)/2}(\bar{\psi}_a\psi_a)^{\frac{N-1}{2}+1} \right)^2.
\end{aligned} \tag{48}$$

The higher interaction term from these shifts,  $(\bar{\psi}_a\psi_a)^{N+1}$ , now requires a further trivial cancellation relation, being that all the subsequent development is unchanged.

### 3.1 Expansion of the model

In the following, the same large fermion mass expansion of the determinant of the last Section is done such as to write down an effective polynomial model for the scalar fields. For the case  $A_1 = 0$ , the contribution of the fermion determinant can be written as:  $Tr \ln \left[ 1 + D \left( B_1 \frac{\xi_1}{d_1} + A_2 \frac{\xi_2}{d_2} \right) \right] + Tr \ln D^{-1}$ , where  $D = \frac{1}{i\bar{\phi} - M^*}$ . The lowest order terms of the zero order derivative expanded model for the auxiliary fields can be written as:

$$\begin{aligned}
\mathcal{V}_{eff}^{(2)} &= \frac{1}{2} \left[ -\frac{1}{d_1} + iTr D^2 \frac{B_1^2}{d_1^2} \right]_{\xi_i=\xi_i^{(0)}} \xi_1^2 - \frac{1}{2} \left[ \frac{1}{d_2} - iTr D^2 \left( \frac{\xi_1^{(0)} B_2}{B_1 d_2} \right)^2 + iTr D \frac{\xi_1^{(0)} B_2^2 d_1}{B_1^3 d_2^2} \right]_{\xi_i=\xi_i^{(0)}} \xi_2^2 \\
&- \frac{1}{2} \left[ \frac{1}{d_3} + \frac{i\delta^2}{\delta\xi_2^2} Tr \log (i\bar{\phi} + \tilde{M}) \right]_{\xi_i=\xi_i^{(0)}} \xi_3^2 + \sum_{i \neq j} c_{i,j} \xi_i(x) \xi_j(x) + \sum_{n_1, n_2, n_3} c_{n_1, n_2, n_3} \xi_1^{n_1}(x) \xi_2^{n_2}(x) \xi_3^{n_3}(x), \\
&(n_1 + n_2 + n_3 \geq 3),
\end{aligned} \tag{49}$$

where the second order term for  $\xi_3$  was not written explicetely because its expression is quite long, and it does not really bring relevant information for the discussion below. This resulting effective potential has a lower degree of symmetry than the one derived in Section II, given by expression (18). The fields can be redefined, in the way it was done in the last section, i.e. by means of  $\xi_i \rightarrow \omega_i G[\xi_i^{(0)}, A_i, B_i]$  with convenient choice of the factors such as to obtain an unique mass term,  $\frac{m^2}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2 + \dots)$ . However, the remaining interactions will have a much lower level of symmetry, i.e.  $c_{1,2} \neq c_{1,3} \neq c_{2,3} \dots$ , or  $c_{3,0,0} \neq c_{0,3,0} \dots$  or  $c_{2,2,0} \neq c_{2,0,2} \dots$  and so on.

Contrarily to the case analyzed in the Section II, the limit of very large condensates, does not yield an effective potential with any apparent symmetry. If one considers the limit of zero condensates, one reaches a non trivial model for the fields  $\xi_1$  and  $\xi_2$  only, independently of the number of auxiliary fields considered. Even in this case it does not have any apparent symmetry. It can be written as:  $V_{eff} = c_{2,1}\xi_1^2 + c_{2,2}\xi_2^2 + c_{12}\xi_1\xi_2 + c_{3,1}\xi_1^3 + c_{3,2}\xi_2^3 + \dots$ , without an usual and satisfactory relation between the resulting masses and effective coupling constants.

Given the two different resulting boson effective models found in Sections II and III, it must be noted that the series presented in the previous section, expression (1), corresponds to a particular case of the more general series given in expression (33). The procedures adopted in both cases were the same and they are based in the introduction of the least number of auxiliary fields with the corresponding shifts. Therefore, by choosing particular terms of the most general



series with particular values of the coupling constants, one might derive a secondary level boson effective model with a higher or lower degree of symmetry very close to the ground state given by a strong enough mean field.

## 4 Series of interactions $\sum_n (\bar{\psi}_a \gamma_\mu \psi_a \cdot \bar{\psi}_b \gamma^\mu \psi_b)^n$

The local limit of an effective fermion model for the case of vector field exchange can be written as:

$$\mathcal{L} = \bar{\psi}_a (i\partial - m_a) \psi_a + \sum_n^N g_{2n} (\bar{\psi}_a \gamma_\mu \psi_a \cdot \bar{\psi}_b \gamma^\mu \psi_b)^n, \quad (50)$$

where  $g_{2n}$  are the effective coupling constants with dimension:  $[g_{2n}] = M^{-d+2n}$ , where  $M$  has dimension of mass,  $m_a$  are the masses for each of the fermion species and the index  $a, b = 1 \dots N_r$  stands for the fermion components. In each bilinear there is a sum over  $a, b$  and the mass term is therefore diagonal.

The auxiliary fields will be introduced by means of the following unity integrals multiplying the generating functional:

$$N' \int \mathcal{D}[\varphi_n] e^{-\frac{i}{2} \int_x (\sum_n^N \varphi_n^2(x) + \varphi_\mu \varphi^\mu)} = 1, \quad (51)$$

where  $N'$  is a normalization constant. The necessary shifts of the a.f. needed to cancel out all the interactions can be made minimal shifts, i.e., the simplest shifts for the minimum number of auxiliary fields which do not introduce Lagrangian terms that were not presented in the original model. For the model of expression (1) the shifts are given by:

$$\begin{aligned} \varphi_\mu^2 &\rightarrow (\varphi_\mu - \beta_1 (\bar{\psi}_a \gamma_\mu \psi_a))^2, \\ \varphi_2^2 &\rightarrow (\varphi_2 - \beta_2 [\bar{\psi}_a \gamma_\mu \psi_a \cdot \bar{\psi}_b \gamma^\mu \psi_b])^2, \\ \varphi_{2m}^2 &\rightarrow (\varphi_{2m} - \beta_{2m} [\bar{\psi}_a \gamma_\mu \psi_a \cdot \bar{\psi}_b \gamma^\mu \psi_b]^m - \alpha_{2m} [\bar{\psi}_a \gamma_\mu \psi_a \cdot \bar{\psi}_b \gamma^\mu \psi_b]^{m-1})^2 \quad (m \geq 2), \end{aligned} \quad (52)$$

where  $\beta_m$  and  $\alpha_m$  are dimensionful parameters that are determined by imposing the corresponding cancelations of all polynomial interactions. Differently from the shifts of Section 2 here there are terms proportional to  $\alpha_{2m}$  which should not appear in the shifts of Section 2 to avoid the appearance of non existing terms in the original fermion interactions. The a.f.  $\varphi_\mu$ ,  $\varphi_2$  and  $\varphi_4$  represent therefore vector fermion-antifermion, two fermion-two antifermion (four fermion) and eight fermion states respectively.

By considering the first four terms ( $N = 4$  and  $m = 2$ ) in the potential (50), the conditions for the cancelations of the polynomial interactions are given by:

$$\begin{aligned} g_8 &= \frac{\beta_4^2}{2}, \\ g_6 &= \beta_4 \alpha_4, \\ g_4 &= -\beta_4 \varphi_4 + \frac{\beta_2^2 + \alpha_4^2}{2}, \\ g_2 &= -\beta_2 \varphi_2 - \alpha_4 \varphi_4 + \frac{\beta_1^2}{2}. \end{aligned} \quad (53)$$

If the parameters  $\beta_n$  might then considered to be functions of different a.f. with the above conditions provided one guarantees that the shifts of the a.f. still have unity Jacobian. In fact all these shifts yield unity Jacobian and different shifts that could cancel out the fermion interactions would introduce other non linearities and the need of extra a.f. or non unity Jacobians. From the relations (53) all the free parameters are fixed unambiguously. They are given by:

$$\begin{aligned}\beta_4 &= \sqrt{2g_8}, \quad \alpha_4 = \frac{g_6}{\beta_4}, \quad \beta_2 = \sqrt{2(g_4 + \beta_4\varphi_4 - 2\alpha_4^2)} \\ \beta_1 &= \sqrt{2(g_2 + \beta_2\varphi_2 + \alpha_4\varphi_4)}\end{aligned}\tag{54}$$

From here on, it will be assumed that the coupling constants values are such that there are well defined solutions for  $\beta_n$  according to the discussion above in Section 2.2. By integrating out fermions it yields the following effective action:

$$S_{eff} = -iTr \log \left( i\cancel{D} - m_a + \beta_1\varphi_\mu\gamma^\mu \right) - \int_x \left( \sum_{n=2}^N \frac{\varphi_n^2}{2} + \frac{\varphi_\mu\varphi^\mu}{2} \right),\tag{55}$$

where  $Tr$  is the traces taken over all the internal indices and integration over space-time. According to expressions (54),  $\beta_1$  depends on all the fields  $\varphi_n$  ( $n \geq 2$ ), i.e.:

$$\beta_1 = \beta_1[\varphi_2, \beta_2] = \beta_1[\varphi_2, \beta_2[\beta_3[\dots[\beta_N]]]]\tag{56}$$

Therefore  $\beta_1$  carries the non linearities of the model.

The gap equations are given by:

$$\begin{aligned}\varphi_\mu &= -i Tr \frac{2\beta_1\gamma_\mu}{i\cancel{D} - m_a + \beta_1\gamma \cdot \varphi}, \\ \varphi_2 &= -i Tr \frac{2(\partial\beta_1/\partial\varphi_2)\varphi^\mu\gamma_\mu}{i\cancel{D} - m_a + \beta_1\gamma \cdot \varphi}, \\ \varphi_4 &= -i Tr \frac{2(\partial\beta_1/\partial\varphi_4)\varphi^\mu\gamma_\mu}{i\cancel{D} - m_a + \beta_1\gamma \cdot \varphi}.\end{aligned}\tag{57}$$

where

$$\frac{\partial\beta_1}{\partial\varphi_2} = \frac{\beta_2}{\beta_1}, \quad \frac{\partial\beta_1}{\partial\varphi_4} = \frac{\alpha_4}{\beta_1} - \frac{\varphi_2\beta_4}{\beta_1^3\beta_2}\tag{58}$$

It will be considered the vector a.f. does not develop a non zero expected value in the vacuum, i.e. the solution for the first gap equation is trivially zero,  $\langle \varphi_\mu \rangle \rightarrow 0$ . This yields necessarily the trivial solutions for the other gap equations.

By expanding the effective action (55) around the minimum within a zero order derivative expansion a complicated structure appears for the interaction between  $\varphi_2$  and  $\varphi_4$ . However, an interesting result is obtained in the case that

$$\frac{\alpha_4}{\beta_1} \gg \frac{\varphi_2\beta_4}{\beta_1\beta_2}$$

for which one may consider:  $\varphi_2\beta_4 \ll \beta_1\beta_2$ . This means weak field  $\varphi_2$  and  $g_8 \ll g_2g_4$ , and therefore  $\partial\beta_1/\partial\varphi_4 \sim \alpha_4/\beta_1$ . In this limit it is possible to write down an interesting form of the effective potential, it yields:

$$\mathcal{V}_I^{eff} = \frac{1}{2} [\varphi_\mu^2 (c_2 + c_{2,1}) + c_2\varphi_2^2 + c_2\varphi_4^2] + V(\varphi_\mu) + \sum_{n=2}^N \sum_m D_{n,m} (\varphi \cdot \gamma)^m \left( \frac{\beta_2}{\beta_1} \varphi_2 + \frac{\alpha_4}{\beta_1} \varphi_4 \right)^n$$

where  $c_2 = 1$ ,  $V(\varphi_\mu)$  depends exclusively on  $\varphi_\mu$  which will not be analysed here,  $D_{n,m}$  are the coefficients of each of the terms of the expansion. The coefficients can be calculated by considering the following quantity  $S_0 = \frac{1}{\gamma \cdot k - m_a}$ , although their explicit shape do not bring any relevant information and thus they are not written explicitly. Next the following redefinition of the a.f. can be considered:

$$\begin{aligned} \varphi_2 &\rightarrow \frac{\beta_2}{\beta_1^3} \varphi_2 \equiv \phi_2, & \varphi_4 &\rightarrow \frac{\alpha_4}{\beta_1^3} \varphi_4 \equiv \phi_4, \\ \varphi_\mu &\rightarrow \beta_1 \varphi_\mu \equiv \phi_\mu. \end{aligned} \quad (59)$$

It yields the following effective potential:

$$\mathcal{V}_{eff} = \frac{m_2^2}{2} \phi_2^2 + \frac{m_4^2}{2} \phi_4^2 + \frac{m_1^2}{2} \phi_\mu^2 + V(\varphi_\mu) + \sum_{m=1} \sum_{n=1} Tr(d_{n,m}) (\gamma_\mu \cdot \phi^\mu)^m (\phi_2 + \phi_4)^n,$$

where:

$$m_2^2 = \frac{\beta_1^6}{\beta_2^2}, \quad m_4^2 = \frac{\beta_1^6}{\alpha_4^2}, \quad m_1^2 = \left( \frac{1}{\beta_1^2} - Tr S_0^2 \gamma_\mu \gamma_\nu \right). \quad (60)$$

and where  $d_{n,m}$  are the coefficients from the expansion.

The important point here is that it appears an approximate symmetry for two a.f.  $\phi_2$  and  $\phi_4$  (59). For instance, if the masses  $m_2^2, m_4^2$  are neglected or very small, the remaining part of the effective potential (60) is invariant under the following transformations:

$$\begin{aligned} \phi_2 &\rightarrow a_2 \phi_2 + a_4 \phi_4 + a_0, \\ \phi_4 &\rightarrow b_2 \phi_2 + b_4 \phi_4 - a_0, \end{aligned} \quad (61)$$

being that

$$a_2 + b_2 = a_4 + b_4 = 1.$$

If one includes the mass terms above, the following transformations preserve the effective potential:

$$\phi_2 \rightarrow a_2 \phi_2 + a_4 \phi_4, \quad \phi_4 \rightarrow b_2 \phi_2 + b_4 \phi_4, \quad (62)$$

being that the following conditions must imposed:

$$b_2^2 = \frac{m_2^2}{m_4^2} (1 - a_2^2), \quad \text{and} \quad a_4^2 = \frac{m_4^2}{m_2^2} (1 - b_4^2). \quad (63)$$

These expressions yield:

$$b_4 = \frac{a_2 \frac{m_4^2}{m_2^2}}{\sqrt{1 + a_2^2 \left(1 + \frac{m_4^4}{m_2^4}\right)}}, \quad \text{and} \quad \frac{m_2^2}{m_4^2} \simeq \frac{g_6^2}{4g_8g_4}, \quad (64)$$

where it has been considered only the leading order for  $\beta_2^2$  (i.e. for  $\beta_2^2 \sim 2g_4$ ) and  $\beta_1^2 \sim 2g_2$ . in the expression for  $m_2^2/m_4^2$  with the masses given in (60).

It is interesting to note that the bosonized effective model (60) corresponds to a model with two massive scalar (2n-fermion composite states) bosons coupled to a massive vector (fermion-antifermion composite state) boson built from the corresponding bilinears. All the scalar boson effective interactions depend necessarily on the vector auxiliary field.

## 5 Summary and conclusions

Three effective fermion models were investigated by means of the auxiliary field method. A minimal procedure was adopted to introduce the minimum number of dynamical auxiliary fields and the minimum number of shifts to produce the desired cancelation of the fermion interactions. This reduces eventual ambiguities in the calculation. In this minimal procedure it was assumed quite strong coupling constants (except the one for the highest order coupling) with respect to (normalized) auxiliary fields that only fluctuates weakly around the ground state, therefore being weak with respect to the condensates. Possible extensions to lift the condition of weak fields were proposed, being that they yield the same final effective boson model and factorization result. The solution of the (coupled) gap equations corresponds to the solution of the first gap equation with however a strong dependence on the coupled expressions for the functions  $\beta_n$ , as presented in the case of the first model with expressions (6). For larger number of fermion components ( $N_r$ ) solutions of the gap equations only can be found in higher dimensions. For the cases in which the gap equations present solutions several conclusions could be drawn. For both models it was found that all the higher order operators and condensates factorize into the lowest order, i.e.  $\langle (\bar{\psi}_a \psi_a)^n \rangle = \langle \bar{\psi}_a \psi_a \rangle^n$ . One exception was found for the case a constant shift in the lowest order auxiliary variable that was considered for the most general series (second one), shift  $A_1$ , expression (35), for  $\xi_1 \rightarrow \xi_1 - B_1 \bar{\psi}_a \psi_a - A_1$ . In this case the higher order condensates do not factorize into the lowest order one. The shift  $A_1$  corresponds to a non trivial overall subtraction of the corresponding lowest order condensate  $\xi_1^{(0)}$ .

As a second step, the fermion determinants of the models were expanded in powers of the (weak) auxiliary fields. These resulting effective models describe composite fermion states and interactions, being therefore related to a previous fermion dynamical model. The resulting polynomial interaction terms were found to have meaningful different structures from each other. By comparing the resulting boson effective models given by expressions (18) (or the limit presented in expression (24)) and (49) it is seen that the former has a more symmetric shape. Furthermore, the field  $\chi_1$  (and, analogously, the field  $\xi_1$ ) can have a different contribution for the overall model from the contribution of the other fields  $\chi_2$  and  $\chi_3$ , and  $\chi_n$  in general, being still more apparent for the case of the second and third models for  $\xi_n$  and  $\varphi_n$  respectively. Therefore it

might happen that only a sector of the resulting boson effective model presents a more specific symmetry instead of the full model. This is the case of the third case analysed in Section 4. The limit of progressively large condensates (progressively weak  $\beta_n$ ) for the model of Section 2 was shown to provide a quite simple effective potential in expression (24). It was found to be invariant under continuous transformations that preserve the length  $\sum_n(\chi_n)$ , and also discrete permutation transformations. This invariance does not come out in the second model for more general series. However in the last case, for the interactions of the form  $\sum_j^N(\bar{\psi}_a\gamma_\mu\psi_a)^{2j}$ , without the formation of vector (or any other) condensates a similar symmetry appeared for the higher order auxiliary fields representing 2 and 4 particle states  $\varphi_2, \varphi_4$ . This last case represents a local and momentum independent limit of a effective potential of a theory of fermions interacting by means of vector field exchange with the basic structure of Quantum Electrodynamics in that limit. Therefore these results suggest that different higher order powers of fermion bilinears might yield boson models for N-fermion states with (approximated) symmetry depending on the terms considered in such series and on the values of the original coupling constants of the fermion model. As discussed in the Introduction, although a renormalization group flow for the first series investigated in Section (2) can yield a full series of the type of Section (3), it is possible to figure out that the terms of the first series keep the approximate resulting symmetry while the other terms of the more general series tend to break it. This scenario might be realistic depending on the resulting relative strength of the coupling constants of the more general series as commented above. The cases of fermions with the corresponding symmetries for the internal quantum numbers (such as SU(2) or SU(3) flavor) were outside the scope of this work. Since the appearance of the approximate symmetry for 2n-fermion states in the bosonized model was obtained without considering a chiral symmetry in the departing model of Section (2) it is concluded that the approximate degeneracy between these multifermion scalar states is not due to a chiral symmetry. In hadron physics, the lightest scalar mesons with similar masses around 1 GeV have seemingly different structures such as quark-antiquark and tetraquark content [26, 27] with similar masses. This approximate degeneracy may correspond to the approximate symmetry of expressions (18) and (24). One might expect however that for the phenomenological coupling constants of the light scalar mesons (comparable to the effective model of Section (2)) chiral symmetry should be a relevant symmetry to be considered. Although the fermion-antifermion channels might be different they must be related to two fermions and tetrafermion states and interactions by crossing or Fierz transformation. Furthermore it is interesting to note that the auxiliary boson fields defined above correspond to a set of states. In (non relativistic) cold atoms there are also different n-particle states (2, 3 or 4 fermion or boson states) that have been observed to have similar energies (binding energies) [24, 25]. Although this is a non relativistic system the resulting symmetry corresponds to the one found in the present work, i.e. a degeneracy between states with different number of particles. Several questions arise such as: which kinds of resulting approximate symmetries is it possible to obtain in the bosonized model by switching on and off particular terms of the original higher order effective potential for a particular structure of fermion bilinears  $\bar{\psi}\Gamma\psi$  (where  $\Gamma$  is one particular operator acting on spinor or other internal space)?

## Acknowledgements

The author thanks short discussions with F.S. Navarra, P. Bedaque and J. Helayel Neto, and partial financial support by CNPq- Brazil and FAPEG-Goiás, Brazil.

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